FINAL: ALGEBRAIC NUMBER THEORY

Date: **4th May 2017**
The total points is **110** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (4+4+4+4+4=20 points) Determine whether the following statements are true or false (No justification needed).
  (a) Let R be a Dedekind domain with fraction field K, L/K a finite separable field extension and R' be the integral closure of R in L. Then R' is a finitely generated R-module.
  (b) Let R be a Dedekind domain with fraction field K, L/K a finite separable extension and R' be the integral closure of R in L. Then every subring of R' containing R is a Dedekind domain.
  (c) Let $\mathcal{L}$ be a lattice in a finite dimensional vector space V. Then $\mathcal{L}$ is free abelian group of finite rank.
  (d) Let $\mathcal{L}$ be a subgroup of a real finite dimensional vector space V such $\mathcal{L}$ is a free abelian group of finite rank. Then $\mathcal{L}$ is a lattice of V.
  (e) Let R be a Dedekind domain. If the class group of R_P for every maximal ideal P of R is trivial then class group of R is trivial.

(2) (5+15=20 points) Define the trace $T_{L/K}(a)$ and the norm $N_{L/K}(a)$ where L/K is a finite field extension and $a \in L$. Let R be a normal domain and an integral domain S be an integral extension of R. Let K and L be fraction fields of R and S respectively such that $[L : K] < \infty$. Show that $T_{L/K}(S) \subset R$ and $N_{L/K}(S) \subset R$.

(3) (5+10+5=20 points) Let R be a Dedekind domain with fraction field K and L/K be finite separable extension. Let R' be the integral closure of R in L. Define the discriminant of R' over R. When $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{5}, \sqrt{7})$, compute all the primes of $\mathbb{Z}$ ramified in R'. Find a unit in the ring of integers of L which not a root of unity.

(4) (15 points) Let K be a number field and R be its ring of integers. Let I be an ideal of R. Show that if the norm of the ideal $N(I)$ is a prime ideal of $\mathbb{Z}$ then I is a prime ideal of R.

(5) (5+10=15 points) Define Legendre symbol. Determine if the primes 43 and 71 are squares modulo the prime 2017.

(6) (5+15=20 points) State Dirichlet's unit theorem. Let p be an odd prime and $[K : \mathbb{Q}] = p$. Let U be the group of units in the ring of integers of K. Show that the torsion subgroup of U has order 2 and rank of U is at least $(p-1)/2$.